Simple Type Theory as a Clausal Theory

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1 Introduction

Deduction modulo is an extension of first-order predicate logic where axioms are replaced by rewrite rules. For instance, the axiom $P \Leftrightarrow (Q \Rightarrow R)$ is replaced by the rule $P \longrightarrow (Q \Rightarrow R)$. These rules define an equivalence relation and, in a proof, a proposition can be replaced by an equivalent one at any time. A presentation of Simple Type Theory in Deduction modulo, called HOL, has been given in [3].

Polarized deduction modulo [1] is an extension of Deduction modulo where possibly different rewrite rules apply to the negative and positive occurrences of atomic propositions. Like any theory expressed in Deduction modulo, Simple Type Theory can be expressed in Polarized deduction modulo. Each rule just needs to be duplicated in a negative and a positive instance.

A rewrite system in Polarized deduction modulo is said to be clausal when negative rules rewrite atomic propositions to clausal propositions and positive rules rewrite atomic propositions to the negation of clausal propositions. This way, clauses rewrite to clauses, which is a useful property in automated theorem proving [2].

We give in this note a presentation of Simple Type Theory as a clausal rewrite system in Polarized deduction modulo, called $HOL^{\pm}$.

This system is does not have the cut elimination property in general but cut elimination holds for sequents well-formed in the language of HOL and, for such sequents, provability in $HOL^{\pm}$ and in HOL are equivalent.

2 Polarized deduction modulo

Definition 1 (Polarized rewrite system). *A* polarized rewrite system *is a triple $\mathcal{R} = \langle \mathcal{E}, \mathcal{R}_-, \mathcal{R}_+ \rangle$ where $\mathcal{E}$ is a set of equations between terms, $\mathcal{R}_-$ and $\mathcal{R}_+$ are sets of rewrite rules whose left hand sides are atomic propositions and right hand sides are arbitrary propositions. The rules of $\mathcal{R}_-$ are called* negative *and those of $\mathcal{R}_+$ are called* positive*.*

Definition 2 (Polarized rewriting). *Let $\mathcal{R} = \langle \mathcal{E}, \mathcal{R}_-, \mathcal{R}_+ \rangle$ be a polarized rewrite system. We define the equivalence relation $=_{\mathcal{E}}$ as the congruence on terms generated by the equations of $\mathcal{E}$. We then define the one step negative and positive rewriting relations $\longrightarrow_-$ and $\longrightarrow_+$ as follows.*

- *If* $t_i =_{\mathcal{E}} t'$ *then* $P(t_1, \ldots, t_i, \ldots, t_n) \longrightarrow_- P(t_1, \ldots, t', \ldots, t_n)$ *and* $P(t_1, \ldots, t_i, \ldots, t_n) \longrightarrow_+ P(t_1, \ldots, t', \ldots, t_n)$.
- *If* $P \longrightarrow A$ *is a rule of* $\mathcal{R}_-$ *and* σ *is a substitution then* $\sigma P \longrightarrow_- \sigma A$. *If* $P \longrightarrow A$ *is a rule of* $\mathcal{R}_+$ *and* σ *is a substitution then* $\sigma P \longrightarrow_+ \sigma A$.
- *If* $A \longrightarrow_+ A'$ *then* $\neg A \longrightarrow_- \neg A'$. *If* $A \longrightarrow_- A'$ *then* $\neg A \longrightarrow_+ \neg A'$.
- *If* $(A \longrightarrow_- A'$ *and* $B = B')$ *or* $(A = A'$ *and* $B \longrightarrow_- B')$, *then* $A \wedge B \longrightarrow_- A' \wedge B'$ *and* $A \vee B \longrightarrow_- A' \vee B'$. *If* $(A \longrightarrow_+ A'$ *and* $B = B')$ *or* $(A = A'$ *and* $B \longrightarrow_+ B')$, *then* $A \wedge B \longrightarrow_+ A' \wedge B'$ *and* $A \vee B \longrightarrow_+ A' \vee B'$.
- *If* $(A \longrightarrow_+ A'$ *and* $B = B')$ *or* $(A = A'$ *and* $B \longrightarrow_- B')$, *then* $A \Rightarrow B \longrightarrow_- A' \Rightarrow B'$. *If* $(A \longrightarrow_- A'$ *and* $B = B')$ *or* $(A = A'$ *and* $B \longrightarrow_+ B')$, *then* $A \Rightarrow B \longrightarrow_+ A' \Rightarrow B'$.
- *If* $A \longrightarrow_- A'$ *then* $\forall x\ A \longrightarrow_- \forall x\ A'$ *and* $\exists x\ A \longrightarrow_- \exists x\ A'$. *If* $A \longrightarrow_+ A'$ *then* $\forall x\ A \longrightarrow_+ \forall x\ A'$ *and* $\exists x\ A \longrightarrow_+ \exists x\ A'$.

We define the sequent one step term rewriting relation $\longrightarrow$ *as follows.*

- *If* $A \longrightarrow_- A'$ *then* $(\Gamma, A \vdash \Delta) \longrightarrow (\Gamma, A' \vdash \Delta)$.
- *If* $A \longrightarrow_+ A'$ *then* $(\Gamma \vdash A, \Delta) \longrightarrow (\Gamma \vdash A', \Delta)$.

As usual, if R is any binary relation, we write R^* for its reflexive-transitive closure. The rules of *Polarized sequent calculus modulo* are those of Fig. 1. Proof checking is decidable when the relations $\longrightarrow_-^*$ and $\longrightarrow_+^*$ are. The usual, non polarized, Deduction modulo can be recovered by taking $\mathcal{R}_- = \mathcal{R}_+$ and predicate logic by taking $\mathcal{E} = \mathcal{R}_- = \mathcal{R}_+ = \varnothing$.

A *theory* is a pair $\langle \mathcal{R}, \mathcal{T} \rangle$ formed with a polarized rewrite system $\mathcal{R}$ and a set of axioms $\mathcal{T}$. We say that the sequent $\Gamma \vdash \Delta$ is *provable in the theory* $\langle \mathcal{R}, \mathcal{T} \rangle$, or that it is *provable in* $\mathcal{T}$ *modulo* $\mathcal{R}$, if there exists a finite subset $\mathcal{T}'$ of $\mathcal{T}$ such that the sequent $\Gamma, \mathcal{T}' \vdash \Delta$ is provable in Polarized sequent calculus modulo $\mathcal{R}$. When $\mathcal{T}$ is empty, we simply say that the sequent $\Gamma \vdash \Delta$ is *provable modulo* $\mathcal{R}$. When $\mathcal{R}$ is empty, we say that the sequent $\Gamma \vdash \Delta$ is *provable in* $\mathcal{T}$ *in predicate logic*.

As discussed in [1], rewriting, in general, has two properties. First, it is oriented and, for instance, the proposition $x \in \mathcal{P}(y)$ rewrites to $\forall z\ (z \in x \Rightarrow z \in y)$, but $\forall z\ (z \in x \Rightarrow z \in y)$ does not rewrite to $x \in \mathcal{P}(y)$. Then, rewriting preserves provability. For instance, the proposition $x \in \mathcal{P}(y)$ rewrites to $\forall z\ (z \in x \Rightarrow z \in y)$ that is provably equivalent. Thus, we can always transform the proposition $x \in \mathcal{P}(y)$ into $\forall z\ (z \in x \Rightarrow z \in y)$ and we never need to backtrack on this operation. When rewriting is polarized, the first property is kept, but not the second. For instance, if we have the negative rule $P \longrightarrow Q$, the sequent $P \vdash P$ can be proved with the axiom rule, but its normal form $Q \vdash P$ cannot.

Definition 3 (Literal, Clausal proposition). *A proposition is a* literal *if it is either atomic of the negation of an atomic proposition. A proposition is* clausal *if it is* $\bot$ *or of the form* $\forall x_1 \ldots \forall x_p\ (L_1 \vee \ldots \vee L_n)$ *where* $L_1, \ldots, L_n$ *are literals and* $x_1, \ldots, x_p$ *variables.*

$$\frac{}{A \vdash B} \text{ axiom if } A \longrightarrow_-^* P, B \longrightarrow_+^* P \text{ and } P \text{ atomic}$$

$$\frac{\Gamma, B \vdash \Delta \quad \Gamma \vdash C, \Delta}{\Gamma \vdash \Delta} \text{ cut if } A \longrightarrow_-^* B, A \longrightarrow_+^* C$$

$$\frac{\Gamma, B, C \vdash \Delta}{\Gamma, A \vdash \Delta} \text{ contr-left if } A \longrightarrow_-^* B, A \longrightarrow_-^* C$$

$$\frac{\Gamma \vdash B, C, \Delta}{\Gamma \vdash A, \Delta} \text{ contr-right if } A \longrightarrow_+^* B, A \longrightarrow_+^* C$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta} \text{ weak-left}$$

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta} \text{ weak-right}$$

$$\frac{}{\Gamma \vdash A, \Delta} \top\text{-right if } A \longrightarrow_+^* \top$$

$$\frac{}{\Gamma, A \vdash \Delta} \bot\text{-left if } A \longrightarrow_-^* \bot$$

$$\frac{\Gamma \vdash B, \Delta}{\Gamma, A \vdash \Delta} \neg\text{-left if } A \longrightarrow_-^* \neg B$$

$$\frac{\Gamma, B \vdash \Delta}{\Gamma \vdash A, \Delta} \neg\text{-right if } A \longrightarrow_+^* \neg B$$

$$\frac{\Gamma, B, C \vdash \Delta}{\Gamma, A \vdash \Delta} \wedge\text{-left if } A \longrightarrow_-^* (B \wedge C)$$

$$\frac{\Gamma \vdash B, \Delta \quad \Gamma \vdash C, \Delta}{\Gamma \vdash A, \Delta} \wedge\text{-right if } A \longrightarrow_+^* (B \wedge C)$$

$$\frac{\Gamma, B \vdash \Delta \quad \Gamma, C \vdash \Delta}{\Gamma, A \vdash \Delta} \vee\text{-left if } A \longrightarrow_-^* (B \vee C)$$

$$\frac{\Gamma \vdash B, C, \Delta}{\Gamma \vdash A, \Delta} \vee\text{-right if } A \longrightarrow_+^* (B \vee C)$$

$$\frac{\Gamma \vdash B, \Delta \quad \Gamma, C \vdash \Delta}{\Gamma, A \vdash \Delta} \Rightarrow\text{-left if } A \longrightarrow_-^* (B \Rightarrow C)$$

$$\frac{\Gamma, B \vdash C, \Delta}{\Gamma \vdash A, \Delta} \Rightarrow\text{-right if } A \longrightarrow_+^* (B \Rightarrow C)$$

$$\frac{\Gamma, C \vdash \Delta}{\Gamma, A \vdash \Delta} \langle x, B, t \rangle \ \forall\text{-left if } A \longrightarrow_-^* \forall x\ B,\ (t/x)B \longrightarrow_-^* C$$

$$\frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A, \Delta} \langle x, B \rangle \ \forall\text{-right if } A \longrightarrow_+^* \forall x\ B,\ x \notin FV(\Gamma \Delta)$$

$$\frac{\Gamma, B \vdash \Delta}{\Gamma, A \vdash \Delta} \langle x, B \rangle \ \exists\text{-left if } A \longrightarrow_-^* \exists x\ B,\ x \notin FV(\Gamma \Delta)$$

$$\frac{\Gamma \vdash C, \Delta}{\Gamma \vdash A, \Delta} \langle x, B, t \rangle \ \exists\text{-right if } A \longrightarrow_+^* \exists x\ B,\ (t/x)B \longrightarrow_+^* C$$

Fig. 1. Polarized sequent calculus modulo

Definition 4 (Clausal rewrite system). *A rewrite system is* clausal *if negative rules rewrite atomic propositions to a clausal propositions and positive rules atomic propositions to negations of clausal propositions.*

3 Equivalence

We want to show that rewrite rules build-in axioms, i.e. that for each rewrite system $\mathcal{R}$, there is a set of axioms $\mathcal{U}_{\mathcal{R}}$ such that a sequent is provable modulo $\mathcal{R}$ if and only if it is provable in $\mathcal{U}_{\mathcal{R}}$ in predicate logic. The set of axioms we wish to consider contains for each equational axiom $t = u$ of $\mathcal{E}$, the universal closure of the proposition $t = u$, for each rule $P \longrightarrow A$ of $\mathcal{R}_-$, the universal closure of the proposition $P \Rightarrow A$, and for each rule $P \longrightarrow A$ of $\mathcal{R}_+$, the universal closure of the proposition $A \Rightarrow P$. A problem is that the language we start with need not contain an equality predicate. Thus, we must first add such a predicate and the axioms of equality and prove that this extension is conservative.

Definition 5 (Compatibility). *Let $\mathcal{R}$ be a polarized rewrite system and $\mathcal{T}$ and $\mathcal{U}$ be two sets of axioms. The theory $\langle \mathcal{R}, \mathcal{T} \rangle$ is* compatible *with $\mathcal{U}$ if*

- *if $A \longrightarrow_-^* B$ in $\mathcal{R}$, then $\vdash A \Rightarrow B$ is provable in $\mathcal{U}$ in predicate logic,*
- *if $A \longrightarrow_+^* B$ in $\mathcal{R}$, then $\vdash B \Rightarrow A$ is provable in $\mathcal{U}$ in predicate logic,*
- *if $A \in \mathcal{T}$, then $\vdash A$ is provable in $\mathcal{U}$ in predicate logic,*
- *if $A \in \mathcal{U}$, then $\vdash A$ is provable in $\mathcal{T}$ modulo $\mathcal{R}$.*

Proposition 1 (Equivalence). *Let $\mathcal{R}$ be a polarized rewrite system and $\mathcal{T}$ and $\mathcal{U}$ be two sets of axioms such that the theory $\langle \mathcal{R}, \mathcal{T} \rangle$ is compatible with $\mathcal{U}$, then a sequent is provable in $\mathcal{T}$ modulo $\mathcal{R}$, if and only if it is provable in $\mathcal{U}$ in predicate logic.*

Proof. If the sequent $\Gamma \vdash \Delta$ is provable in $\mathcal{U}$ in predicate logic, there exists a finite subset $\mathcal{U}'$ of $\mathcal{U}$ such that $\Gamma, \mathcal{U}' \vdash \Delta$ is provable in predicate logic and hence modulo $\mathcal{R}$. Each U_i in $\mathcal{U}'$ is provable in $\mathcal{T}$ modulo $\mathcal{R}$, thus, for each U_i, there exists a finite subset $\mathcal{T}_i'$ of $\mathcal{T}$ such that $\mathcal{T}_i' \vdash U_i$ is provable modulo $\mathcal{R}$. Let $\mathcal{T}'$ be the union of all the $\mathcal{T}_i$'s. Using the cut rule, we build a proof of $\Gamma, \mathcal{T}' \vdash \Delta$ modulo $\mathcal{R}$. Thus, the sequent $\Gamma \vdash \Delta$ is provable in $\mathcal{T}$ modulo $\mathcal{R}$.

The converse is a simple induction over proof structure.

Definition 6. *Let $\mathcal{R}$ be a polarized rewrite system. Let $\mathcal{A}_{\mathcal{R}}$ be the set of axioms containing*

- *for each pair of propositions A and B such that $A \longrightarrow_-^* B$, the universal closure of $A \Rightarrow B$,*
- *for each pair of propositions A and B such that $A \longrightarrow_+^* B$, the universal closure of $B \Rightarrow A$.*

Proposition 2. *Let $\mathcal{R}$ be a polarized rewrite system and $\mathcal{T}$ be a set of axioms. Then, the theory $\langle \mathcal{R}, \mathcal{T} \rangle$ and the set of axioms $\mathcal{A}_{\mathcal{R}} \cup \mathcal{T}$ are compatible.*

Proof. If $A \longrightarrow_{-}^{*} B$ in $\mathcal{R}$, then the universal closure of $A \Rightarrow B$ is an element of $\mathcal{A}_{\mathcal{R}}$. Thus, the sequent $\vdash A \Rightarrow B$ is provable in $\mathcal{A}_{\mathcal{R}} \cup \mathcal{T}$ in predicate logic. If $A \longrightarrow_{+}^{*} B$ in $\mathcal{R}$, then the universal closure of $B \Rightarrow A$ is an element of $\mathcal{A}_{\mathcal{R}}$. Thus, the sequent $\vdash B \Rightarrow A$ is provable in $\mathcal{A}_{\mathcal{R}} \cup \mathcal{T}$ in predicate logic. If $A \in \mathcal{T}$, then $A \in \mathcal{A}_{\mathcal{R}} \cup \mathcal{T}$ and thus the sequent $\vdash A$ is provable in $\mathcal{A}_{\mathcal{R}} \cup \mathcal{T}$ in predicate logic.

Conversely, if $A \in \mathcal{T}$, then the sequent $\vdash A$ is provable in $\mathcal{T}$ modulo $\mathcal{R}$ and if $A \in \mathcal{A}_{\mathcal{R}}$, then the sequent $\vdash A$ is provable in $\mathcal{T}$ modulo $\mathcal{R}$.

Definition 7 (Model). *Let $\mathcal{R}$ be a polarized rewrite system and $\mathcal{T}$ be a set of axioms, a* model *of the theory $\langle \mathcal{R}, \mathcal{T} \rangle$ is a model of the set of axioms $\mathcal{A}_{\mathcal{R}} \cup \mathcal{T}$.*

Proposition 3 (Soundness and completeness). *A sequent $\Gamma \vdash \Delta$ is provable in $\mathcal{T}$ modulo $\mathcal{R}$ if and only if valid in all models of $\langle \mathcal{R}, \mathcal{T} \rangle$.*

Proof. By Propositions 1 and 2, the sequent $\Gamma \vdash \Delta$ is provable in $\mathcal{T}$ modulo $\mathcal{R}$ if and only if it is provable in $\mathcal{A}_{\mathcal{R}} \cup \mathcal{T}$. By the soundness and completeness theorem of predicate logic it is provable in $\mathcal{A}_{\mathcal{R}} \cup \mathcal{T}$ if and only if it is valid in all models of $\mathcal{A}_{\mathcal{R}} \cup \mathcal{T}$, i.e. in all models of $\langle \mathcal{R}, \mathcal{T} \rangle$.

Definition 8 (Equality model). *Let $\mathcal{R}$ be a polarized rewrite system and $\mathcal{T}$ be a set of axioms. An* equality model *of $\langle \mathcal{R}, \mathcal{T} \rangle$ is a model where if $t =_{\mathcal{E}} u$ then for all ϕ, $[\![t]\!]_\phi = [\![u]\!]_\phi$.*

Proposition 4 (Soundness and completeness for equality models). *A sequent $\Gamma \vdash \Delta$ is provable in $\mathcal{T}$ modulo $\mathcal{R}$ if and only if valid in all equality models of $\langle \mathcal{R}, \mathcal{T} \rangle$.*

Proof. All we need to prove is that for each model $\mathcal{M}$ of $\langle \mathcal{R}, \mathcal{T} \rangle$ we can build an equality model of $\langle \mathcal{R}, \mathcal{T} \rangle$. Let $\mathcal{M}$ be a model of $\langle \mathcal{R}, \mathcal{T} \rangle$. We write $\mathcal{M}_T$ for the domain of $\mathcal{M}$ of sort T, $\hat{f}$ for the interpretation of the function symbol f and $\hat{P}$ for the interpretation of the predicate symbol P. For each sort T, we define the relation $\sim_T$ on the elements of $\mathcal{M}_T$, by $a \sim_T b$ if and only if there exists two terms t and u of sort T and a valuation ϕ such that $t =_{\mathcal{E}} u$, $a = [\![t]\!]_\phi$ and $b = [\![u]\!]_\phi$. This relation is obviously an equivalence relation and it is compatible with the interpretation of all the function symbols. To prove that it is compatible with the denotation of the predicate symbols, we remark that if $t =_{\mathcal{E}} u$ then $P(t_1, ..., t, ..., t_n) \longrightarrow_{-} P(t_1, ..., u, ..., t_n)$ and $P(t_1, ..., t, ..., t_n) \longrightarrow_{+} P(t_1, ..., u, ..., t_n)$, thus the proposition $P(t_1, ..., t, ..., t_n) \Leftrightarrow P(t_1, ..., u, ..., t_n)$ is provable modulo $\mathcal{R}$ and thus valid in $\mathcal{M}$. We finally define a model $\mathcal{M}'$ by taking $\mathcal{M}'_T = \mathcal{M}_T / \equiv_T$ and by interpreting the function symbol f by the function $\hat{f}/\equiv$ and the predicate symbol f by the function $\hat{P}/\equiv$. The propositions valid in the models $\mathcal{M}$ and $\mathcal{M}'$ are the same.

Definition 9. *Let $\mathcal{L}$ be a language containing an equality predicate in each sort. Let $\mathcal{R}$ be a polarized rewrite system in $\mathcal{L}$. Let $\mathcal{U}_{\mathcal{R}}$ be the set of axioms containing*

- *the axioms of equality for $\mathcal{L}$,*

– *for each equational axiom $t = u$ of $\mathcal{E}$, the universal closure of the proposition $t = u$,*
– *for each rule $P \longrightarrow A$ of $\mathcal{R}_-$, the universal closure of the proposition $P \Rightarrow A$,*
– *for each rule $P \longrightarrow A$ of $\mathcal{R}_+$, the universal closure of the proposition $A \Rightarrow P$.*

Proposition 5. *Let $\mathcal{L}$ be a language containing an equality predicate in each sort. Let $\mathcal{E}q$ be the axioms of equality for $\mathcal{L}$. Let $\mathcal{R}$ be a polarized rewrite system in $\mathcal{L}$. Then, the theory $\langle \mathcal{R}, \mathcal{E}q \rangle$ and the set of axioms $\mathcal{U}_\mathcal{R}$ are compatible.*

Proof. It is routine to check that if $A \longrightarrow_-^* B$ in $\mathcal{R}$, then the sequent $\vdash A \Rightarrow B$ is provable in $\mathcal{U}_\mathcal{R}$ in predicate logic, and if $A \longrightarrow_+^* B$ in $\mathcal{R}$, then the sequent $\vdash B \Rightarrow A$ is provable in $\mathcal{U}_\mathcal{R}$ in predicate logic. If A is an axiom of $\mathcal{E}q$, then it is an axiom of $\mathcal{U}_\mathcal{R}$, hence the sequent $\vdash A$ is provable in $\mathcal{U}_\mathcal{R}$ in predicate logic.

Conversely, we check, considering each of the four cases, that if $A \in \mathcal{U}_\mathcal{R}$, then the sequent $\vdash A$ is provable in $\mathcal{E}q$ modulo $\mathcal{R}$.

Proposition 6. *Let $\mathcal{R}$ be a polarized rewrite system in a language $\mathcal{L}$. Let $\mathcal{L}'$ be the language obtained by adding an equality symbol in each sort of $\mathcal{L}$. Let $\mathcal{E}q$ be the axioms of equality for $\mathcal{L}'$. Then, the theory $\langle \mathcal{R}, \mathcal{E}q \rangle$ is a conservative extension of $\mathcal{R}$, i.e. a sequent $\Gamma \vdash \Delta$ of $\mathcal{L}$ is provable modulo $\mathcal{R}$ if and only if it is provable in $\mathcal{E}q$ modulo $\mathcal{R}$.*

Proof. An equality model of $\mathcal{R}$ extends to an equality model of $\langle \mathcal{R}, \mathcal{E}q \rangle$ by interpreting equality by equality.

Remark that this proof would not go through if we did not consider equality models. Indeed if $t =_\mathcal{E} u$, then $t = t \longrightarrow_- t = u$ and if $t = u$ were not valid in the model, it would not be a model of the proposition $t = t \Rightarrow t = u$.

Proposition 7. *Let $\mathcal{L}$ be a language and $\mathcal{R}$ be a polarized rewrite system in $\mathcal{L}$. Let $\mathcal{L}'$ be the language obtained by adding an equality symbol in each sort of $\mathcal{L}$. Then, a sequent $\Gamma \vdash \Delta$ of $\mathcal{L}$ is provable modulo $\mathcal{R}$ if and only if it is provable in $\mathcal{U}_\mathcal{R}$.*

Proof. Let $\mathcal{E}q$ be the axioms of equality for $\mathcal{L}'$. By Proposition 6, the sequent $\Gamma \vdash \Delta$ is provable modulo $\mathcal{R}$ if and only if it is provable in $\mathcal{E}q$ modulo $\mathcal{R}$ and by Propositions 1 and 5 it is provable in $\mathcal{E}q$ modulo $\mathcal{R}$ if and only if it is provable in $\mathcal{U}_\mathcal{R}$.

4 Simple Type Theory as a clausal rewrite system

A presentation of Simple Type Theory in non polarized deduction modulo has been given in [3]. To define it in polarized deduction modulo we just duplicate each rule. We also consider an extension of the system presented in [3] with rules expressing the existence of a non surjective injection *Succ* of type $\iota \rightarrow \iota$, that allow to prove the "axiom" of infinity.

Definition 10 (The theory *HOL*).

The sorts are simple types*, inductively defined by*

- *ι and o are sorts,*
- *if T and U are sorts then $T \to U$ is a sort.*

As usual, we write $T_1 \to \ldots \to T_n \to U$ for $T_1 \to (\ldots \to (T_n \to U)\ldots)$. The language contains

- *for each pair of sorts T, U, a constant $K_{T,U}$ of sort $T \to U \to T$,*
- *for each triple of sorts T, U, V, a constant $S_{T,U,V}$ of sort $(T \to U \to V) \to (T \to U) \to T \to V$,*
- *a constant $\dot{\vee}$ or sort $o \to o \to o$,*
- *a constant $\dot{\neg}$ or sort $o \to o$,*
- *for each sort T, a constant $\dot{\forall}_T$ of sort $(T \to o) \to o$,*
- *a constant 0 of sort ι, two constants Succ and Pred of sort $\iota \to \iota$, and a constant Null of sort $\iota \to o$,*
- *for each pair of sorts T, U, a function symbol $\alpha_{T,U}$ of rank $\langle T \to U, T, U\rangle$,*
- *a predicate symbol ε of rank $\langle o \rangle$.*

As usual, we write $(t\ u)$ for $\alpha_{T,U}(t, u)$ and $(t\ u_1\ \ldots\ u_n)$ for $(\ldots(t\ u_1)\ldots u_n)$. The rewrite rules are

$$(K_{T,U}\ x\ y) =_{\mathcal{E}} x$$

$$(S_{T,U,V}\ x\ y\ z) =_{\mathcal{E}} (x\ z\ (y\ z))$$

$$(Pred\ (Succ\ x)) =_{\mathcal{E}} x$$

$$\begin{array}{rlrl}
\varepsilon(x\ \dot{\vee}\ y) & \longrightarrow_- (\varepsilon(x) \vee \varepsilon(y)) & \varepsilon(x\ \dot{\vee}\ y) & \longrightarrow_+ (\varepsilon(x) \vee \varepsilon(y)) \\
\varepsilon(\dot{\neg}\ x) & \longrightarrow_- \neg\varepsilon(x) & \varepsilon(\dot{\neg}\ x) & \longrightarrow_+ \neg\varepsilon(x) \\
\varepsilon(\dot{\forall}_T\ x) & \longrightarrow_- \forall y\ \varepsilon(x\ y) & \varepsilon(\dot{\forall}_T\ x) & \longrightarrow_+ \forall y\ \varepsilon(x\ y) \\
\varepsilon(Null\ (S\ x)) & \longrightarrow_- \bot & \varepsilon(Null\ (S\ x)) & \longrightarrow_+ \bot \\
\varepsilon(Null\ 0) & \longrightarrow_- \top & \varepsilon(Null\ 0) & \longrightarrow_+ \top
\end{array}$$

The theory HOL is not clausal. We now define a clausal theory $HOL^{\pm}$ and prove it is equivalent to HOL.

Definition 11 (The theory $HOL^{\pm}$). *The sorts are the same as those of HOL. The symbols are the same as those of HOL and, for each sort T, a function symbol H_T of sort $(T \to o) \to T$. The rewrite rules are*

$$(K_{T,U}\ x\ y) =_{\mathcal{E}} x$$

$$(S_{T,U,V}\ x\ y\ z) =_{\mathcal{E}} (x\ z\ (y\ z))$$

$$(Pred\ (Succ\ x)) =_{\mathcal{E}} x$$

$$\begin{array}{rlrl}
\varepsilon(x\ \dot{\vee}\ y) & \longrightarrow_- (\varepsilon(x) \vee \varepsilon(y)) & \varepsilon(x\ \dot{\vee}\ y) & \longrightarrow_+ \neg\neg\varepsilon(x) \\
 & & \varepsilon(x\ \dot{\vee}\ y) & \longrightarrow_+ \neg\neg\varepsilon(y) \\
\varepsilon(\dot{\neg}\ x) & \longrightarrow_- \neg\varepsilon(x) & \varepsilon(\dot{\neg}\ x) & \longrightarrow_+ \neg\varepsilon(x) \\
\varepsilon(\dot{\forall}_T\ x) & \longrightarrow_- \forall y\ \varepsilon(x\ y) & \varepsilon(\dot{\forall}_T\ x) & \longrightarrow_+ \neg\neg\varepsilon(x\ H_T(x)) \\
\varepsilon(Null\ (S\ x)) & \longrightarrow_- \bot & & \\
 & & \varepsilon(Null\ 0) & \longrightarrow_+ \neg\bot
\end{array}$$

Proposition 8. *If a sequent, containing no occurrence of the symbols H_T, has a proof in $HOL^{\pm}$, then it has a proof in HOL.*

Proof. Using Proposition 7, all we need to prove is that the theory $\mathcal{U}_{HOL^{\pm}}$ is a conservative extension of $\mathcal{U}_{HOL}$.

The theories $\mathcal{U}_{HOL^{\pm}}$ and $\mathcal{U}_{HOL}$ differ on three points. First, the theory $\mathcal{U}_{HOL^{\pm}}$ contains the axioms $\forall x \forall y\ (\neg\neg\varepsilon(x) \Rightarrow \varepsilon(x \dot{\vee} y))$ and $\forall x \forall y\ (\neg\neg\varepsilon(y) \Rightarrow \varepsilon(x \dot{\vee} y))$ while the theory $\mathcal{U}_{HOL}$ contains the axiom $\forall x \forall y\ ((\varepsilon(x) \vee \varepsilon(y)) \Rightarrow \varepsilon(x \dot{\vee} y))$. But the conjunction of the two axioms of $\mathcal{U}_{HOL^{\pm}}$ is equivalent to that of $\mathcal{U}_{HOL}$.

Second, the theory $\mathcal{U}_{HOL}$ contains two axioms $\varepsilon(\mathit{Null}\ 0) \Rightarrow \top$ and $\forall x\ (\bot \Rightarrow \varepsilon(\mathit{Null}\ (S\ x)))$. But these axioms are trivially provable in predicate logic and they can be eliminated.

Third, the theory $\mathcal{U}_{HOL^{\pm}}$ contains the axiom $\forall x\ (\neg\neg\varepsilon(x\ H_T(x)) \Rightarrow \varepsilon(\dot{\forall}_T\ x))$ and the axioms of equality for the symbols H_T and the theory $\mathcal{U}_{HOL}$ the axiom $\forall x\ ((\forall y\ \varepsilon(x\ y)) \Rightarrow \varepsilon(\dot{\forall}_T\ x))$. But the axiom of $\mathcal{U}_{HOL^{\pm}}$ is equivalent to the Skolemization of that of $\mathcal{U}_{HOL}$.

Thus, using Skolem theorem for classical logic with equality, we get that $\mathcal{U}_{HOL^{\pm}}$ is a conservative extension of $\mathcal{U}_{HOL}$.

Trivially, if a sequent $\Gamma \vdash \Delta$, containing no occurrence of the symbols H_T, has a cut free proof in $HOL^{\pm}$, it has a proof in $HOL^{\pm}$ and thus it has a proof in HOL. Using the cut elimination theorem for HOL, we get that it has a cut free proof in HOL. We now want to prove the converse, i.e. that if a sequent, containing no occurrence of the symbols H_T, has a cut free proof in HOL, it has a cut free proof in $HOL^{\pm}$.

Proposition 9. *If $(\Gamma \vdash \Delta) \longrightarrow^* (\Gamma' \vdash \Delta')$ and $\Gamma' \vdash \Delta'$ has a cut free proof modulo $\mathcal{R}$ then $\Gamma \vdash \Delta$ has a cut free proof modulo $\mathcal{R}$ of the same size.*

Proof. By induction over proof structure.

Proposition 10. *If a sequent containing no occurrence of the symbols H_T, has a cut free proof in HOL, it has a cut free proof in $HOL^{\pm}$.*

Proof. Let $\Gamma \vdash \Delta$ be a sequent that has a cut free proof in HOL. By induction on the size of this proof, we build a cut free proof of this sequent in $HOL^{\pm}$. We give only two cases.

- If the proof has the form

$$\dfrac{\dfrac{\pi}{\Gamma \vdash B, C, \Delta}}{\Gamma \vdash A, \Delta}\ \vee\text{-right}$$

  with $A \longrightarrow_{+}^{HOL} (B \vee C)$, then either $A = (B' \vee C')$ or A is atomic.
  In the first case we have $B' \longrightarrow_{+}^{HOL} B$, $C' \longrightarrow_{+}^{HOL} C$. By Proposition 9, the sequent $\Gamma \vdash B', C', \Delta$ has a cut free proof of the same size, by induction hypothesis it has a cut free proof in $HOL^{\pm}$ and we conclude with the $\vee$-right rule.

In the second, consider a reduction sequence from A to $B \vee C$ and in this reduction sequence, the last atomic proposition A' and its successor $B' \vee C'$. We have $A \longrightarrow_{+}^{HOL*} A' \longrightarrow_{+}^{HOL} (B' \vee C') \longrightarrow_{+}^{HOL*} (B \vee C)$. As A' is atomic and $A' \longrightarrow_{+}^{HOL} (B \vee C)$, we have $A' = \varepsilon(t \mathbin{\dot\vee} u)$, $B' = \varepsilon(t)$, and $C' = \varepsilon(u)$. As $(\varepsilon(t) \vee \varepsilon(u)) \longrightarrow_{+}^{HOL*} (B \vee C)$, we have $\varepsilon(t) \longrightarrow_{+}^{HOL*} B$ and $\varepsilon(u) \longrightarrow_{+}^{HOL*} C$. By Proposition 9, the sequent $\Gamma \vdash \varepsilon(t), \varepsilon(u), \Delta$ has a cut free proof of the same size in HOL and by induction hypothesis, it has a cut free proof in $HOL^{\pm}$. As A and $\varepsilon(t \mathbin{\dot\vee} u)$ are atomic and $A \longrightarrow_{+}^{HOL*} \varepsilon(t \mathbin{\dot\vee} u)$, we have $A \longrightarrow_{+}^{HOL^{\pm}*} \varepsilon(t \mathbin{\dot\vee} u)$. Then, $\varepsilon(t \mathbin{\dot\vee} u) \longrightarrow_{+}^{HOL^{\pm}} \neg\neg\varepsilon(t)$ and $\varepsilon(t \mathbin{\dot\vee} u) \longrightarrow_{+}^{HOL^{\pm}} \neg\neg\varepsilon(u)$. Thus, $A \longrightarrow_{+}^{HOL^{\pm}*} \neg\neg\varepsilon(t)$ and $A \longrightarrow_{+}^{HOL^{\pm}*} \neg\neg\varepsilon(u)$. We build a cut free proof of $\Gamma \vdash A, \Delta$ in $HOL^{\pm}$ with the rules contraction-right, $\neg$-right, and $\neg$-left and the proof of $\Gamma \vdash \varepsilon(t), \varepsilon(u), \Delta$.

– If the proof has the form

$$\dfrac{\dfrac{\pi}{\Gamma \vdash B, \Delta}}{\Gamma \vdash A, \Delta}\ \forall\text{-right}$$

with $A \longrightarrow_{+}^{HOL} \forall x\ B$, then either $A = \forall x\ B'$ or A is atomic.
In the first case we have $B' \longrightarrow_{+}^{HOL} B$. By Proposition 9, the sequent $\Gamma \vdash B', \Delta$ has a cut free proof of the same size, by induction hypothesis it has a cut free proof in $HOL^{\pm}$ and we conclude with the $\forall$-right rule.
In the second, consider a reduction sequence from A to $\forall x\ B$ and in this reduction sequence, the last atomic proposition A' and its successor $\forall x\ B'$. We have $A \longrightarrow_{+}^{HOL*} A' \longrightarrow_{+}^{HOL} \forall x\ B' \longrightarrow_{+}^{HOL*} \forall x\ B$. As A' is atomic and $A' \longrightarrow_{+}^{HOL} \forall x\ B$, we have $A' = \varepsilon(\dot\forall_T\ t)$ and $B' = \varepsilon(t\ x)$. As $\forall x\ \varepsilon(t\ x) \longrightarrow_{+}^{HOL*} \forall x\ B$, we have $\varepsilon(t\ x) \longrightarrow_{+}^{HOL*} B$. By Proposition 9, the sequent $\Gamma \vdash \varepsilon(t\ x), \Delta$ has a cut free proof of the same size in HOL and by induction hypothesis, it has a cut free proof in $HOL^{\pm}$. By substituting the term $H_T(t)$ for the variable x in this proof, we get a proof of the sequent $\Gamma \vdash \varepsilon(t\ H_T(t))$ in $HOL^{\pm}$. As A and $\varepsilon(\dot\forall_T\ t)$ are atomic and $A \longrightarrow_{+}^{HOL*} \varepsilon(\dot\forall_T\ t)$, we have $A \longrightarrow_{+}^{HOL^{\pm}*} \varepsilon(\dot\forall_T\ t)$. Then, $\varepsilon(\dot\forall_T\ t) \longrightarrow_{+}^{HOL^{\pm}} \neg\neg\varepsilon(t\ H_T(t))$. Thus, $A \longrightarrow_{+}^{HOL^{\pm}*} \neg\neg\varepsilon(t\ H_T(t))$. We build a cut free proof of $\Gamma \vdash A, \Delta$ in $HOL^{\pm}$ with the rules $\neg$-right and $\neg$-left and the proof of $\Gamma \vdash \varepsilon(t\ H_T(t)), \Delta$.

Proposition 11. *For a sequent containing no occurrence of the symbols H_T the following conditions are equivalent*

1. *the sequent has a proof in $HOL^{\pm}$,*
2. *it has a proof in HOL,*
3. *it has a cut free proof in HOL,*
4. *it has a cut free proof in $HOL^{\pm}$.*

Proof. *1.* $\Rightarrow$ *2.* is Proposition 8, *2.* $\Rightarrow$ *3.* is the cut elimination for HOL [6, 7, 5] (see also [4]), *3.* $\Rightarrow$ *4.* is Proposition 10, *4.* $\Rightarrow$ *1.* is trivial.

Notice that $HOL^{\pm}$ does not have the cut elimination property in general. For instance, the sequent $\varepsilon(x\ H_T(x)) \vdash \forall y\ \varepsilon(x\ y)$ has a proof with a cut (on

$\varepsilon(\dot{\forall}_T\ x))$ but no cut free proof. Yet, for sequents well-formed in the language of HOL (i.e. containing no symbols H_T), the cut elimination property holds and provability is equivalent to provability in HOL.

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